

Quaternions

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A quaternion is a four-dimensional complex number that can be used to represent the orientation of a rigid body or coordinate frame in three-dimensional space. An arbitrary orientation of frame B relative to frame A can be achieved through a rotation of angle θ around an axis ${}^A\hat{\mathbf{r}}$ defined in frame A . This is represented graphically in figure 1 where the mutually orthogonal unit vectors $\hat{\mathbf{x}}_A, \hat{\mathbf{y}}_A$ and $\hat{\mathbf{z}}_A$, and $\hat{\mathbf{x}}_B, \hat{\mathbf{y}}_B$ and $\hat{\mathbf{z}}_B$ define the principle axis of coordinate frames A and B respectively. The quaternion describing this orientation, ${}^A_B\hat{\mathbf{q}}$, is defined by equation (1) where r_x, r_y and r_z define the components of the unit vector ${}^A\hat{\mathbf{r}}$ in the x, y and z axes of frame A respectively. A notation system of leading super-scripts and sub-scripts adopted from Craig [1] is used to denote the relative frames of orientations and vectors. A leading sub-script denotes the frame being described and a leading super-script denotes the frame this is with reference to. For example, ${}^A_B\hat{\mathbf{q}}$ describes the orientation of frame B relative to frame A and ${}^A\hat{\mathbf{r}}$ is a vector described in frame A . Quaternion arithmetic often requires that a quaternion describing an orientation is first normalised. It is therefore conventional for all quaternions describing an orientation to be of unit length.

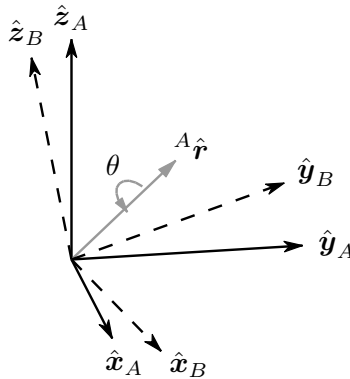


Figure 1: The orientation of frame B is achieved by a rotation, from alignment with frame A , of angle θ around the axis ${}^A\hat{\mathbf{r}}$.

$${}^A_B\hat{\mathbf{q}} = [q_0 \ q_1 \ q_2 \ q_3] = [\cos\frac{\theta}{2} \ -r_x\sin\frac{\theta}{2} \ -r_y\sin\frac{\theta}{2} \ -r_z\sin\frac{\theta}{2}] \quad (1)$$

The quaternion conjugate, denoted by $*$, can be used to swap the relative frames described by an orientation. For example, ${}^B_A\hat{\mathbf{q}}$ is the conjugate of ${}^A_B\hat{\mathbf{q}}$ and describes the orientation of frame A relative to frame B . The conjugate of ${}^A_B\hat{\mathbf{q}}$ is defined by equation (2).

$${}^A_B\hat{\mathbf{q}}^* = {}^B_A\hat{\mathbf{q}} = [q_0 \quad -q_1 \quad -q_2 \quad -q_3] \quad (2)$$

The quaternion product, denoted by \otimes , can be used to define compound orientations. For example, for two orientations described by ${}^A_B\hat{\mathbf{q}}$ and ${}^B_C\hat{\mathbf{q}}$, the compounded orientation ${}^A_C\hat{\mathbf{q}}$ can be defined by equation (3).

$${}^A_C\hat{\mathbf{q}} = {}^B_C\hat{\mathbf{q}} \otimes {}^A_B\hat{\mathbf{q}} \quad (3)$$

For two quaternions, \mathbf{a} and \mathbf{b} , the quaternion product can be determined using the Hamilton rule and defined as equation (4). A quaternion product is not commutative; that is, $\mathbf{a} \otimes \mathbf{b} \neq \mathbf{b} \otimes \mathbf{a}$.

$$\begin{aligned} \mathbf{a} \otimes \mathbf{b} &= [a_0 \quad a_1 \quad a_2 \quad a_3] \otimes [b_0 \quad b_1 \quad b_2 \quad b_3] \\ &= \begin{bmatrix} a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3 \\ a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2 \\ a_0b_2 - a_1b_3 + a_2b_0 + a_3b_1 \\ a_0b_3 + a_1b_2 - a_2b_1 + a_3b_0 \end{bmatrix}^T \end{aligned} \quad (4)$$

A three dimensional vector can be rotated by a quaternion using the relationship described in equation (5) [2]. ${}^A\mathbf{v}$ and ${}^B\mathbf{v}$ are the same vector described in frame A and frame B respectively where each vector contains a 0 inserted as the first element to make them 4 element row vectors.

$${}^B\mathbf{v} = {}^A_B\hat{\mathbf{q}} \otimes {}^A\mathbf{v} \otimes {}^A_B\hat{\mathbf{q}}^* \quad (5)$$

The orientation described by ${}^A_B\hat{\mathbf{q}}$ can be represented as the rotation matrix ${}^A_B\mathbf{R}$ defined by equation (6) [2].

$${}^A_B\mathbf{R} = \begin{bmatrix} 2q_0^2 - 1 + 2q_1^2 & 2(q_1q_2 + q_0q_3) & 2(q_1q_3 - q_0q_2) \\ 2(q_1q_2 - q_0q_3) & 2q_0^2 - 1 + 2q_2^2 & 2(q_2q_3 + q_0q_1) \\ 2(q_1q_3 + q_0q_2) & 2(q_2q_3 - q_0q_1) & 2q_0^2 - 1 + 2q_3^2 \end{bmatrix} \quad (6)$$

A quaternion may be obtain from a rotation matrix using the inverse of the relationships defined in (6); however, in some practical applications an available rotation matrix may not be orthogonal and so a more robust method is preferred. Bar-Itzhack provides a method [3] to extract the optimal, ‘best fit’ quaternion from an imprecise and non-orthogonal rotation matrix. The method requires the construction of the symmetric 4 by 4 matrix \mathbf{K} (equation (7)) where r_{mn} corresponds to the element of the m^{th} row and n^{th} column of ${}^A_B\mathbf{R}$. The optimal quaternion, ${}^A_B\hat{\mathbf{q}}$, is found as the normalised Eigen vector corresponding to the maximum Eigen value of \mathbf{K} . This is defined by equation (8) where v_0 to v_3 define the elements of the normalised Eigen vector.

$$\mathbf{K} = \frac{1}{3} \begin{bmatrix} r_{11} - r_{22} - r_{33} & r_{21} + r_{12} & r_{31} + r_{13} & r_{23} - r_{32} \\ r_{21} + r_{12} & r_{22} - r_{11} - r_{33} & r_{32} + r_{23} & r_{31} - r_{13} \\ r_{31} + r_{13} & r_{32} + r_{23} & r_{33} - r_{11} - r_{22} & r_{12} - r_{21} \\ r_{23} - r_{32} & r_{31} - r_{13} & r_{12} - r_{21} & r_{11} + r_{22} + r_{33} \end{bmatrix} \quad (7)$$

$${}^A_B\hat{\mathbf{q}} = [v_3 \ v_0 \ v_1 \ v_2] \quad (8)$$

The ZYX Euler angles ϕ , θ and ψ describe an orientation of frame B achieved by the sequential rotations, from alignment with frame A , of ψ around $\hat{\mathbf{z}}_B$, θ around $\hat{\mathbf{y}}_B$, and ϕ around $\hat{\mathbf{x}}_B$. This Euler angle representation of ${}^A_B\hat{\mathbf{q}}$ can be calculated [4] using equations (9) to (11).

$$\phi = \text{atan2} (2(q_2q_3 - q_0q_1), 2q_0^2 - 1 + 2q_3^2) \quad (9)$$

$$\theta = -\arctan \left(\frac{2(q_1q_3 + q_0q_2)}{\sqrt{1 - (2q_1q_3 + 2q_0q_2)^2}} \right) \quad (10)$$

$$\psi = \text{atan2} (2(q_1q_2 - q_0q_3), 2q_0^2 - 1 + 2q_1^2) \quad (11)$$

References

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